1. (10 points) **(How good is greedy for Vertex Cover)** This will drive down the reason we study other algorithms for set cover even though in general we know that greedy is optimal. There could be a large family of instances which have structure where we can outperform greedy.

   (a) (10 points) Construct an example where the greedy algorithm has an approximation ratio of $\Omega(\log n)$ for the vertex cover problem where there are n vertices in the graph.

   Solution:

   Proof. $1 + 1 = 2$. □

2. (25 points) **(Finishing the Set Cover Rounding Proof)** We'd left the final parts of the proof as homework. You'll now complete this.

   (a) (10 points) We showed the following two properties which our rounding algorithm satisfies (if we repeated the randomized rounding experiment for $T = 2 \ln n$ steps: (i) the expected cost is $2 \ln n \mathsf{Opt}$ where Opt is the cost of the optimal LP fractional solution, and (ii) the probability with which all elements are covered is at least $1 - \frac{1}{n}$. Show that there with some constant probability, we will find a solution which has cost at most $O(1) \ln n$ and also covers all the elements. (Hint: Use Markov's inequality and the union bound)

   Solution:

   Proof. $1 + 1 = 2$. □

   (b) (10 points) Now if instead of running our rounding $T = 2 \ln n$ times, if we had run it a different number (say, $\ln n + C \ln \ln n$) of times. Then try to optimize the parameters and show that we will compute, with some non-trivial probability of $\Omega(\frac{1}{\ln n})$, a solution where the cost is $(\ln n + O(\ln \ln n)) \mathsf{Opt}$ and all elements are covered.

   Solution:

   Proof. $1 + 1 = 2$. □

   (c) (5 points) Finally boost the success probability above by repeating this algorithm some number of times. Roughly how many times do you need to run to get probability of failure to be e^{-n}?

   Solution:

   Proof. $1 + 1 = 2$. □

3. (20 points) **(Integrality Gap for Robust Min-Sum-Set-Cover)** Consider the generalization of min-sum-set-cover where the cover time of an element is defined to be the first time when the element is covered K times, for a given parameter K. We will now show that the natural LP has a large integrality gap for this instance.

   (a) (10 points) Write the natural LP for this problem.

   Solution:

   Proof. $1 + 1 = 2$. □

   (b) (10 points) Consider the following instance, and show that it has a large integrality gap. The universe of elements $U = \{e_1, e_2, \ldots, e_l\}$. The sets are $\mathcal{S} = \{S_1 \equiv \{e_1, e_2, \ldots, e_l\}, S_2 \equiv \{e_1, e_2, \ldots, e_l\}, \ldots, S_n \equiv \{e_1, e_2, \ldots, e_l\}, S_{n+1} = \{e_1\}, S_{n+2} = \{e_2\}, \ldots, S_{n+l} = \{e_l\}\}$. Suppose the coverage requirement $K = (n + 1)$. Show that we can set values of l and n so that the LP solution and integral solutions have a large gap. For this, you need to exhibit some fractional solution of low cost and show that all integral solutions have much larger cost.

   Solution:

   Proof. $1 + 1 = 2$. □

4. (30 points) **(Structure of a fractional optimum for the vertex cover LP relaxation)** Recall in class that we wrote down an integer linear program of two variable inequalities (one per edge) such that a feasible 0-1 solution is a vertex cover. Let VC denote this integer linear program, and let LPVC denote the vertex relaxation. Let x^* an optimum solution to LPVC and let V_0, V_1, V_h be the 3 vertex sets of the graph as discussed in class.

   (a) (5 points) Show that $N(V_0) = V_1$.

   Solution:

   Proof. $1 + 1 = 2$. □

   (b) (10 points) Show that the value of x^* is $|V_1| - |V_0| + \frac{|V_h|}{2}$.

   Solution:

   Proof. $1 + 1 = 2$. □

   (c) (5 points) Show that all the corner points of the polytope are half-integral.

   Solution:

Proof. $1 + 1 = 2$. □

(d) (10 points) Use the above arguments to compute the minimum vertex cover of a tree.

Solution:

Proof. $1 + 1 = 2$. □