1. (25 points) **Familiarity with Euclidean Norms.** In the following exercise, you will familiarize yourself with Euclidean metrics. For any n points $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ where each $\mathbf{x}_i \in \mathbb{R}^d$, define $d(\mathbf{x}_i, \mathbf{x}_j) = \|\mathbf{x}_i - \mathbf{x}_j\|_p$. Here, $\|\mathbf{x}\|_p = \left(\sum_{\ell=1}^{d}(x_l)^p\right)^{1/p}$ for any $\mathbf{x} = (x_1, x_2, \ldots, x_d)^\intercal$ is any vector in $\mathbb{R}^d$ and $p \geq 1$.

   (a) (5 points) Show that (X, d) is a metric space for any $p = 1$ and $p = 2$. In fact, it is a metric space for all $p \geq 1$, but you don't need to prove this.

   Solution:

   Proof. $1 + 1 = 2$. □

   (b) (5 points) Consider the 4-point metric space formed by $d(1,2) = 1, d(1,3) = 2, d(1,4) = 1, d(2,3) = 1, d(2,4) = 2, d(3,4) = 1$. Of course, $d(i,i) = 0$ and $d(i,j) = d(j,i)$ for all $i, j \in \{1,2,3,4\}$ hold. Does this embed into ℓ_1 metrics? If so, how many dimensions do you need in your embedding?

   Solution:

   Proof. $1 + 1 = 2$. □

   (c) (10 points) Consider the same 4-point metric space as above. Does this embed into ℓ_2 metrics? If so, how many dimensions do you need in your embedding? Conclude that ℓ_1 metrics do not embed isometrically into ℓ_2 metrics.

   Solution:

   Proof. $1 + 1 = 2$. □

   (d) (5 points) For any vector $\mathbf{x}$, show that $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n}\,\|\mathbf{x}\|_2$. Using this, show that any finite ℓ_1 metric trivially embeds into an ℓ_2 metric with distortion at most $\sqrt{n}$.

2. (20 points) **Finding the Best Embeddings.** Here, we see algorithmic problems involving embeddings. Indeed, given a metric (X, d), we want to see the best possible embedding into ℓ_2 metrics with minimum distortion. Show that this can be formulated as an SDP.

   Solution:

   Proof. $1 + 1 = 2$. □

3. (30 points) **Cut Metrics and ℓ_1 metrics.** Recall that given a graph $G = (V, E)$, a cut metric $\delta_S(\cdot, \cdot)$ is defined for $S \subseteq V$, and is of the form $\delta_S(u, v) = 1$ if and only if

$S \cap \{u, v\} = 1$, and $\delta_S(u, v) = 0$ otherwise. Also recall that an ℓ_1 metric associates a vector $f(u) \in \mathbb{R}^d$ for all $u \in V$.

(a) (10 points) Suppose the ℓ_1 metric is in 1 dimension, i.e., it is a line. Then show that you can obtain weights $w_S \geq 0$ for all $S \subseteq V$ such that, for all $u \in V, v \in V$, the distance $\|f(u) - f(v)\|_1$ can be expressed as $\sum_{S \subseteq V} w_S \delta_S(u, v)$. (Hint: consider the line embedding and assign non-zero weights only to the subsets which arise as prefixes on this line.)

Solution:

Proof. $1 + 1 = 2$. □

(b) (5 points) Extend the above proof for an ℓ_1 embedding f in dimension d (Hint: just separately do this over all the co-ordinates).

Solution:

Proof. $1 + 1 = 2$. □

(c) (15 points) Using the above idea, show that if we have an ℓ_1 metric $d(\cdot, \cdot)$ which has sparsity[1] $\frac{\sum_{(u,v) \in E} d(u,v)}{\sum_{u \in V, v \in V} d(u,v)} = \lambda$, then we can actually find a cut $(S, \bar{S})$ such that $\frac{|E(S,\bar{S})|}{|S||\bar{S}|} \leq \lambda$. This completes the $O(\log n)$ approximation to sparsest cut using metric embeddings discussed in class.

4. (0 points) **Difficulty Level.** How difficult was this homework? How much time would you have spent on these questions?

Solution:

Proof. $1 + 1 = 2$. □

[1]Here the denominator sums over all pairs (u, v) without double counting any pair. In other words, we are summing over all the edges of the complete graph K_n.