1. (25 points) **(Probabilistic Method)** Probabilistic method refers to the technique of proving the existence of an object by constructing a random process underwhich the desired object is output with non-zero probability. This may sound like a very round-about way of proving things, but is really a more analytics version of the Piegeon-Hole principle.

   In this exercise, we prove the existence of expander graphs as used in the AKS sorting networks. We work with bipartite graphs $G = (L, R, E)$ where $L$ and $R$ are the left and right sets of vertices, satisfying $|L| = |R| = n$; and $E \subseteq L \times R$ is the set of edges between $L$ and $R$. Such a graph is said to be $\epsilon$-left-expanding if for every set $S \subseteq L$ satisfying $|S| \geq \epsilon |L|$,

$$|\mathcal{N}(S) := \{r \in R \mid \exists l \in S : (l, r) \in E\}| \geq (1 - \epsilon) |R| .$$

   Our random process picks $d$ random perfect matchings and sets $E$ to be the union of these perfect matchings.

   (a) (2 points) Prove that the random process outputs a graph where each vertex has degree at most $d$.

   **Solution:**

   *Proof.* $1 + 1 = 2$. □

   (b) (10 points) Let us fix a set $S \subseteq L$ satisfying $|S| \geq \epsilon |L|$, and a set $T \subset R$ satisfying $|T| < (1 - \epsilon) |R|$. Show that the probability that $\mathcal{N}(S) \subseteq T$ is at most $(1 - \epsilon)^{\epsilon n d}$.

   **Solution:**

   *Proof.* $1 + 1 = 2$. □

   (c) (10 points) Above, we have shown that there is no fixed $S, T$ that is a counter-example to the $\epsilon$-left-expanding property of our random graph. To complete the argument, we simply take union bound over all sets $S, T$. Show that for $d$ large enough in terms of $\epsilon$ (but independent of $n$), part (b) holds for every set $S, T$ satisfying the size constraints with probability at least $1 - \exp(-n)$. This proves that the output of the random process is an expander with probability exponentially close to 1.

   **Solution:**

   *Proof.* $1 + 1 = 2$. □

   (d) (3 points) Define $\epsilon$-right-expanding similarly (but with left and right transposed) and further call $G$ $\epsilon$-expanding if it is both left and right expanding. What is the probabilty that a graph $G$ output from the above process is $\epsilon$-expanding?

2. (20 points) **(Different Graph Sparsifier)** In class, you saw one kind of graph sparsifiers, called *spectral sparsifiers*. In this homework, you'll design another kind which approximately preserves pairwise distances in a graph while retaining very few edges. In the following, let $G = (V, E)$ denote a undirected, unweighted graph on $n$ vertices and $m$ edges where $m$ is large, say $\Omega(n^2)$.

   (a) (10 points) Construct a subgraph $H$ in the following manner: randomly sample each vertex $v$ as a "hub" with probability $p$ (i.e., add $v$ to $S$ w.p $p$). Then construct a subgraph $H_v$ which preserves shortest paths to all other vertices from $v$. Using $\cup_{v \in S} H_v$ as a backbone (of course, we may need to add other edges to $H_v$), construct $H$ so that all distances are preserved up to an additive 2. (Hint: break up the graph into high and low degree vertices depending on $p$, and try to use the fact that high-degree vertices are likely to be neighboring sampled hubs.) Optimize for $p$ so that the number of edges in $H$ is at most $O(n^{1.5}\text{polylog}(n))$.

   **Solution:**

   *Proof.* $1 + 1 = 2$. □

   (b) (10 points) Suppose the graph $H$ we construct need not be a subgraph of $G$ (i.e., we can introduce new vertices), and moreover, is allowed to have weights (non-negative) on edges. Show that by adapting the above solution to reach hubs from both end points, we can bring down size of $H$ to $O(n^{4/3}\text{polylog}(n))$ edges but now all distances are only preserved upto additive 4.(Hint: now you can easily preserve distances between hubs by just adding a complete graph with the shortest path distances as the weights.)

   **Solution:**

   *Proof.* $1 + 1 = 2$. □