CS6841: Advanced Algorithms **IIT Madras, Spring 2016**
Lecture #8: Introduction to LP January 22, 2016
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1 Linear Programming

Linear program is an optimization problem over n variables $x_1, x_2...x_n$ with linear constraints and a linear objective function which we aim to maximize or minimize. As an example, consider the following.
Suppose we have n items, each item has u_i grams of protein/kg and it's cost is c_i Rs/kg. Objective is to minimize the cost wherein there is a requirement of B grams of protein. It can be modeled as below.

$$Min \sum c_i x_i$$

$$\sum u_i c_i \leq B$$

$$x_i \geq 0 \quad \forall i = 0, 1, 2...$$

There are various forms of LP. Two most common ones are **general form** and **equational**(**standard**) **form**. The general form of LP is given as,

$$Min \ c^T x$$

$$Ax \geq B$$

where A is a $m * n$ matrix where each row in A corresponds to a given constraint and each column is for one of the n variables. B is a $m \times 1$ column matrix with $B_i = b_i$. x is a $n \times 1$ column vector of n variables and c is the column vector such that $c_i = c_i$.

$$A = \begin{bmatrix} \cdots & \cdots & \cdots & a_1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & a_2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & a_3 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & a_m & \cdots & \cdots & \cdots \end{bmatrix}$$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}; \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}; \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

such that

$$\begin{bmatrix} \cdots & \cdots & \cdots & a_1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & a_2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & a_3 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & a_m & \cdots & \cdots & \cdots \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \geq \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

such that when we expand this we get back,

$$a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n \geq b_1$$

$$a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n \geq b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 \cdots + a_{mn}x_n \geq b_m$$

There are other ways to capture LP. For example, the objective function

$$Max\ c^T x$$

can be written as

$$Min\ \ -c^T x$$

And the two constraints given below are equivalent.

$$\sum a_{ij}x_j \geq b \Longleftrightarrow -\sum a_{ij}x_j \leq -b$$

Therefore upper bound constraints are equivalent to lower bound constraints. Similarly,

$$\sum a_{ij}x_j = b \Longleftrightarrow \sum a_{ij}x_j \leq b\ \ AND\ \ \sum a_{ij}x_j \geq b$$

We can even write inequality constraints in terms of equality constraints by adding "slack" variables. One such example is shown below. Here s_i is the slack variable.

$$a_i x \leq b \Leftrightarrow a_i x + s_i = b\ \ where\ \ s_i \geq 0$$

In general form, the $x's$ can be unconstrained. That is x can take positive or negative values. If we want to restrict our x such that all of them are non negative, we can do so by replacing each x with $x^+ - x^-$ where $x^+ \geq 0$ and $x^- \geq 0$.
To understand better we can verify it with a couple of examples.Suppose $x = 5$ in general form. Then it can be written such that $x^+ = 5$ and $x^- = 0$. Suppose $x = -2$ in general form. Then, $x^+ = 0$ and $x^- = 2$.
This brings us to another form of writing LP which is **equational form**. It is written as follows.

$$Min\ c^T x$$

$$Ax = B$$

$$x \geq 0$$

The equational form and general form of LP are equivalent. Trivially every equational form is also a general form. The other way is also true. That is, we can write every general form constraint using the above said techniques. i.e., every constraint in general form which is of the form

$$a_1x_1 + a_2x_2 \cdots + a_nx_n \geq b$$

can be written as

$$a_1x_1 + a_2x_2 \cdots + a_nx_n - s = b\ AND\ s \geq 0$$

For every variable in the general form replace x_i with $x_i^+ - x_i^-$

Why LP??

1. **Generality** - Many optimization problems are LPs or are very well approximated by LPs

2. **Efficient** - In practice, LPs can be solved efficiently.

How to solve LPs

There are several ways of solving an LP. Simplex is a classic one which explores the basic feasible solutions(further referred to as BFS) and improves the quantity we are trying to maximize(or minimize) at every step of the way. We'll see what a basic feasible solution means shortly. Before that let's set some basic concepts straight.

We view LPs in equational form and find optimal solution to the problem "if one exists" or report infeasibility. That is we find optimal solution for LP of the form

$$Min\ c^T x$$

$$Ax = b$$

$$x \geq 0$$

For example consider the set of constraints

$$x_1 + 2x_2 = 3$$

$$x_1 + 7x_5 = 7$$

$$0.5(x_1 + 2x_2) + 0.3(x_1 + 7x_5) = 3 * 0.5 + 7 * 0.3 + 1$$

Clearly, the above doesn't have a solution as the constraints are inconsistent. Thus we gave an example of how LPs can also be infeasible.

Now for LPs which are feasible, let's look into the way to find basic feasible solution, which are a special kind of feasible solutions. The definition of BFS is explained as we go along. Given the matrix A in equational form, assume that rows of A are linearly independent. If the rows are linearly dependent, some constraints are redundant and can be eliminated. We can use Gaussian elimination technique to check that rows of A are linearly independent. So we have

$$rank(A) = \#\ rows\ \ =\ \ m$$

Among n variables, choose m of them. Let's call this set B. That is, $B \subset [n]$ and $|B| = m$. Let the matrix A_B be an $m \times m$ matrix whose columns are picked from the matrix A be defined such that each column in A_B corresponds to the variable in B. For example, if $B = x_1, x_3, x_5 \ldots$ then, the submatrix A_B is given by,

$$A_B = \begin{bmatrix} \vdots & \vdots & \vdots & \\ 1 & 3 & 5 & \ldots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

where each of the columns 1,3 and 5 are picked from the matrix A. Now solve for

$$A_B X_B = b$$

where X_B contains variables only from B. Also assuming that A_B's columns are linearly independent, A_B would have a full rank and hence invertible. Then X_B would have a unique solution,

$$\boxed{X_B = {A_B}^{-1} b} \tag{8.1}$$

We can extend X_B to include all n variables such that we set $x_i = 0$ for variables which are not in B. This vector X is called a **basic solution**. Note that X may have some negative numbers so it might not satisfy non negativity constraints. If X had all positive x_i's, then we call that basic solution a **basic feasible solution or BFS**. We call B the basis. The algorithm to find BFS is as given below.

$$\boxed{\begin{array}{l} \textit{Enumerate all } B = \binom{n}{m}, \textit{ such that the corresponding columns in } A \textit{ are linearly independent} \\ Set X_B = {A_B}^{-1}b, extend\ to\ X, setting\ everything\ else\ to\ 0 \\ \textit{Output the best solution} \end{array}}$$

Why does this algorithm even work? What if there exists a solution to LP, but no BFS exists? What if the optimum value is something other than the BFS we found? Thankfully we have a theorem which answers our questions. It's stated and proved below.

Theorem 8.1. *For any LP in equational form, one of the following holds.*

1. *The LP is infeasible*
2. *The LP has unbounded optimum*
3. *The LP has an optimal solution x^* which is a BFS*

Proof. Suppose the LP is feasible and has a bounded optimum solution. Let x^* be the optimal solution with fewest number of non-zero components. Let P be the non-zero support in x^*. i.e,

$$P = \{i|\tilde{x_i} > 0\}$$

$[n] - P$ is all zero in x^*. There are two cases to consider.

1. The columns corresponding to the indices in P are **linearly independent**. i.e., the columns of A_P are independent. We know that A has rank m. Therefore size of P is at most m. We can extend P to $\tilde{P}$ such that $|\tilde{P}| = m$ and columns of $A_{\tilde{P}}$ are independent. We can think of $\tilde{P}$ as B such that
$$A_{\tilde{P}}X_{\tilde{P}} = b \Longleftrightarrow A_B X_B = b$$
is got from BFS algorithm. We know that $A_P X_P^* = b$(we assumed x^* is the optimal solution). From this we have $A_{\tilde{P}}X_{\tilde{P}}^* = b$ which we got from $\tilde{P}$ which we get by adding some 0 variables back to P. We claim that $X_{\tilde{P}} = X_{\tilde{P}}^*$ because $A_{\tilde{P}}$ is a full rank matrix($|\tilde{P}| = m$) and hence a unique solution exists.

2. The columns of A_P are **linearly dependent**. Then there exists some coefficients w_i where $i \in P$ such that
$$\sum_{i \in P} w_i A_i = 0$$
By setting w_i to zero for all $i \notin P$, we get w such that $Aw = 0$. We now want to show that x^* doesn't have most zeros of all the optimal solutions. Suppose $c^T w = 0$. Let $\tilde{x} = x^* - \epsilon w$. Then
$$A\tilde{x} = Ax^* - A\epsilon w = 0$$
Then
$$c^T\tilde{x} = c^T x^* - c^T \epsilon w = c^T c^*$$
We'll assume that w has some positive entries. If all are negative, we can set $\tilde{x}$ such that we add ϵw to x^*. So as we increase ϵ, we are decreasing some of the positive entries in $\tilde{x}$ without

changing the objective function. And at some some point if one of the coordinates become zero, then $\tilde{x}$ will have one fewer non zero variable than x^*, contradicting to the fact that x^* has the fewest non zero variables.
Now suppose $c^T w > 0$. Consider $\tilde{x} = x^* - \epsilon w$. Then

$$A\tilde{x} = Ax^* - A\epsilon w = ax^*$$

Then

$$c^T \tilde{x} = c^T x^* - c^T \epsilon w$$

$\tilde{x} \geq 0$ if ϵ is sufficiently small. But by the above equation, we can see that

$$c^T \tilde{x} < c^T x^*$$

Which implies that x^* is not the optimal solution, contrary to our assumption. Similarly if $c^T w < 0$, we can just negate all the entries of w to reduce to the case $c^T w > 0$.

□

Duality of linear programs

Consider the following LP.

$$LP_1 = min\ x_1 + 2x_2 + 4x_3$$

$$x_1 + x_2 + 2x_3 = 5$$

$$2x_1 + x_2 + 3x_3 = 8$$

$$x_1, x_2, x_3 \geq 0$$

Let's ask ourselves a question here. How small can OPT be for LP_1? We can get this easily by comparing the given constraints with the objective function.

$$Objective = x_1 + 2x_2 + 4x_3 \geq x_1 + x_2 + 2x_3 = 5 \implies OPT(LP_1) \geq 5$$

Another important question is using the constraints of LP_1 what is the best provable bound you can show for $OPT(LP_1)$? Suppose we have variables which represents a multiple of each term in the constraints. Let's call them y_1 and y_2 respectively for first and second constraint. Then,

$$x_1(y_1 + 2y_2) + x_2(y_1 + y_2) + x_3(2y_1 + 3y_3) = 5y_1 + 8y_2$$

$$y_1 + 2y_2 \leq 1$$

$$y_1 + y_2 \leq 2$$

$$2y_1 + 3y_3 \leq 4$$

And the problem of optimizing LP_1 reduces to maximizing $5y_1 + 8y_2$. That is,

$$OPT(LP_1) \geq 5y_1 + 8y_2$$

This is nothing but another LP, let's call this LP_2, which can be written as follows.

$$LP_2 = max\ 5y_1 + 8y_2$$

$$y_1 + 2y_2 \le 1$$
$$y_1 + y_2 \le 2$$
$$2y_1 + 3y_3 \le 4$$

We can see that any feasible solution to LP_2 is a bound on OPT(LP_1). LP_1 is called a **primal** and LP_2 is called the **dual** of the primal LP.

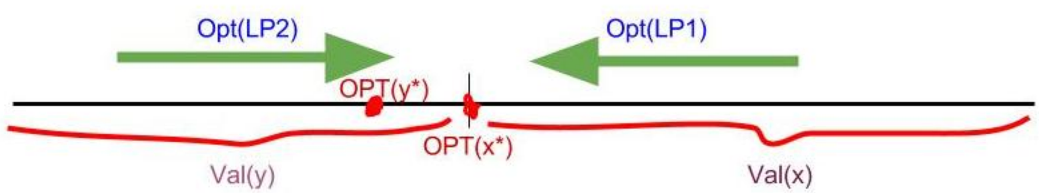
Duality Theorem

The Duality Theorem tells us about the optimal values taken by a primal LP and its dual. Let LP_1 be a primal LP and LP_2 be it's dual.

Theorem 8.2. Weak Duality Theorem :
Let $LP_1 = min\{c^T x | Ax \ge b, x \ge 0\}$ *and* $LP_2 = max\{b^T y | Ay \le c, y \ge 0\}$. *If* x *is a feasible solution for* LP_1 *and* y *is a feasible solution for* LP_2*, then,*

$$value(x, LP_1) \ge value(y, LP_2)$$

Pictorially,



We can also say that if LP_1 is unbounded then LP_2 is infeasible and if LP_2 is unbounded, LP_1 is infeasible.

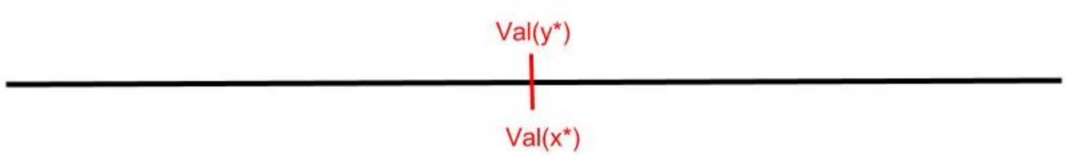
Strong duality theorem says more about the optimal solutions about primal-dual pair.

Theorem 8.3. Strong Duality Theorem :
Let $LP_1 = min\{c^T x | Ax \ge b, x \ge 0\}$ *and* $LP_2 = max\{b^T y | Ay \le c, y \ge 0\}$. *If* LP_1 *and* LP_2 *are both feasible and bounded, then*

$$value(x^*) = value(y^*)$$

where x^* *and* y^* *are optimal values of* LP_1 *and* LP_2 *respectively.*

That is both have same optimal solutions when both are bounded and feasible. Pictorially,



Below is the table of possibilities for the feasibility of LP_1 and LP_2

Dual \ Primal	Infeasible	Unbounded	Feasible and Bounded
Infeasible	Yes	Yes	No
Unbounded	Yes	No	No
Feasible and Bounded	No	No	Yes, Equality

Note that the case where both primal and dual are infeasible is more of a corner case. An example where both primal and dual are infeasible is,

$$max\ 2x_1 - x_2$$

$$x_1 - x_2 \leq 1$$

$$-x_1 + x_2 \leq -2$$

$$x_1 \geq 0, x_2 \geq 0$$

You can verify that both this LP and its dual is infeasible.