## Spectral Sparsification

## 1 Recall

## 2 Laplacian

The Laplacian Matrix of a weighted graph $\mathrm{G}=(\mathrm{V}, \mathrm{E}, \mathrm{w}), w: E \rightarrow R^{+}$,
Laplacian in quadratic form:

$$
x^{T} L x=\sum_{e=(a, b)}\left(x_{a}-x_{b}\right)^{2} w_{a, b}
$$

Adjancy matrix A:
$a(i, j)=1$ iff $(\mathrm{i}, \mathrm{j}) \in \mathrm{E}$.
Laplacian $\mathrm{L}=\mathrm{D}$ - A where D is diagonal matrix of degrees.

## 3 Resistor networks

This is a physical model of a graph in which we treat every edge as a resistor. If the graph is unweighted, assume that each resistor has resistance 1. If an edge e has weight $w_{e}$, give the corresponding resistor resistance $r_{e}=\frac{1}{w_{e}}$. Because when the weight of an edge is very small, the edge is barely there, so it should correspond to very high resistance.Having no edge corresponds to having a resistor of infinite resistance. Letters $\mathrm{a}, \mathrm{b}$ and c for the names of vertices, v for voltages.
Ohms law:

$$
V=I R
$$

That is, the potential drop across a resistor ( V ) is equal to the current flowing over the resistor (I) times the resistance (R). To apply this in a graph, we will define for each edge $(\mathrm{a}, \mathrm{b})$ the current flowing from u to a to be $\mathrm{i}(\mathrm{a}, \mathrm{b})$. As this is a directed quantity, we define

$$
i(b, a)=-i(a, b)
$$

let $\mathrm{v} \in R^{V}$ be a vector of potentials (voltages) at vertices. Given these potentials, we can figure out how much current flows on each edge by the formula.

$$
i(a, b)=\frac{(v(a)-v(b))}{r_{a, b}} w_{a, b}(v(a)-v(b))
$$

That is, we adopt the convention that current flows from high voltage to low voltage. I write this equation in matrix form. The one complication is that each edge comes up twice in i . So, to treat i as a vector each edge show up exactly once as $(\mathrm{a}, \mathrm{b})$ when $a \in b$. define the
signed edge-vertex adjacency matrix of the graph $U$ to be the matrix with rows indexed by edges and columns indexed by vertices such that

$$
U((a, b), c)=\left\{\begin{array}{cl}
1 & \text { if } \mathrm{a}=\mathrm{b} \\
-1 & \text { if } \mathrm{b}=\mathrm{c} \\
0 & \text { otherwise }
\end{array}\right.
$$

write the row of U corresponding to edge ( $\mathrm{a}, \mathrm{b}$ ) as $\delta_{a}-\delta_{b}$. W be the diagonal matrix with rows and columns indexed by edges and the weights of edges on the diagonals.

$$
i=W U v .
$$

Let $i_{\text {ext }} \in R_{V}$ denote the external currents, where $i_{e x t}(a)$ is the amount of current entering the graph through node $a$. We then have

$$
i_{e x t}(a)=\sum_{b:(a, b) \in E} i(a, b)
$$

in matrix form,

$$
i_{e x t}=U^{T} i=U^{T} W U v .
$$

in matrix form

$$
L=U^{T} W U
$$

is the Laplacian

$$
L=\sum_{(u, v) \in E} w_{u, v}\left(\delta_{u}-\delta_{v}\right)\left(\delta_{u}-\delta_{v}\right)^{T}
$$

the nodes a for which

$$
i_{e x t}(a) \neq 0
$$

as being boundary nodes.

$$
i_{e x t}(a)=L v
$$

for the internal nodes.
If the graph is unweighted and a is an internal node, then the ath row of this equation is

$$
0=\left(\delta_{a}^{T} L\right) v=\sum_{(a, b) \in E}(v(a)-v(b))=d(a) v(a)-\sum_{(a, b) \in E} v(b)
$$

i.e

$$
v(a)=\frac{\sum_{(a, b) \in E} v(b)}{d(a)}
$$

## 4 Effective resistance

Effective resistance of the whole network between a and b. Consider electrical flow that sends one unit of current into node a and removes one unit of current from node b, measure the potential difference between a and b that is required to realize this current, is said to be effective resistance between a and b, notation $R_{e f f}(a, b)$.
the potential difference between a and b in a flow of one unit of current from a to b :

$$
R_{e f f}=v(a)-v(b)=\left(\delta_{a}-\delta_{b}\right)^{T} L^{+}\left(\delta_{a}-\delta_{b}\right)
$$

## 5 Sparsification

1 : Sparsify a graph whil maintaining structure.
2 : Sample edge with probability $\propto \frac{1}{R_{e f f}} 3:$ For all graph G we'll get approximation(This will involved effective resistance) containing only $\mathrm{O}(\mathrm{n} \log (\mathrm{n}))$ edges. Approximation :: a graph H to be an $\epsilon$-approximation of a graph G if

$$
(1-\epsilon) L_{H} \preccurlyeq L_{G} \preccurlyeq(1+\epsilon) L_{H} .
$$

as graphs that approximate each other have a lot in common. For example, 1. the effective resistance between all pairs of vertices are similar in the two graphs, 2. the eigenvalues of the graphs are similar, 3. the boundaries of all sets are similar, and 4. the solutions of linear equations in the two matrices are similar.

## 6 Positive semi definite matrice(PSD)

1. $A \succcurlyeq 0 \Longleftrightarrow \lambda_{i}(A) \geq 02$. Partial Ordering on matrices
$A \succcurlyeq B \Longleftrightarrow A-B \succcurlyeq 0$
2. Spectral Sparsify :

$$
(1-\epsilon) L_{H} \preccurlyeq L_{G} \preccurlyeq(1+\epsilon) L_{H} .
$$

here G is our original graph.
4. $A \succcurlyeq 0 \Longrightarrow A^{T}=\sum \lambda_{T} e_{i} e_{i}^{T}$

Note that $\lambda_{i} \geq 0$ are eigen values and $e_{i}$ are corresponind eigen vectors of A.

## $7 \quad$ Sparsifying Complete graphs

$\mathrm{G}=K_{n}$ what are the eigen values of complete graphs? $\lambda(G)$ :
$\lambda_{1}=1$.
and any other eigen values are as $\lambda_{j}=1-\frac{1}{n-1}$ where n is very large number. $L_{G} \overline{1}=0$
Approximation : any sparse graph s.t $\lambda_{1}=1$ and $\lambda_{j} \geq 1-\epsilon$
A very good expander
Ramanujan Expander $\equiv$ complete graph. $L_{G}$ denotes for laplacian for a graph G. $S \subseteq V$

$$
x_{s}(i)= \begin{cases}1 & \text { if } \mathrm{i} \in \mathrm{~S} \\ 0 & \text { otherwise }\end{cases}
$$

$\mathrm{L}=\mathrm{D}-\mathrm{A}$

$$
x^{T} L x=\sum_{e=(i, j)}\left(x_{s}(i)-x_{i}(j)^{2}\right) w_{a, b}
$$

Refferences ::
Spectral Graph theory by Dan Spielman.
OutLine of the Algorithm::

1. Input a graph G

If $R_{a b}$ denotes for effective resistance between ( $\mathrm{a}, \mathrm{b}$ )
2. Output will be a graph H and H will be a sparsifier with probability $>0.75$.
3. $q_{a, b}=w_{a, b} \cdot R_{a, b}$
4. $p_{a, b}=\frac{c(l o g n)}{\epsilon^{2}}$
5. insert ( $\mathrm{a}, \mathrm{b}$ ) with probability $p_{a, b}$ and $w_{H}(a, b)=\frac{w_{a, b}}{p_{a, b}}$
6. output graph H.Output graph still be a weighted graph $w_{H}(a, b)=\frac{1}{p_{a, b}}$.

Our original Laplacian

$$
\begin{aligned}
L_{G} & =\sum_{(a, b)} w_{a, b} L_{a, b} \\
L_{H} & =\sum_{(a, b)} z_{a, b} L_{a, b}
\end{aligned}
$$

where

$$
x^{T} L_{G} x=\sum_{(a, b)} w_{a, b} L_{a, b}
$$

$$
x^{T} L_{G} x=\sum_{(a, b)}\left(x_{a}-x_{b}\right)^{2}=\sum x_{T} L_{a, b} x
$$

$$
z_{a, b}=\left\{\begin{array}{cl}
\frac{w_{a, b}}{p_{a, b}} & \text { with probability } p_{a, b} \\
0 & \text { otherwise }
\end{array}\right.
$$

$$
\begin{equation*}
E\left[L_{H}\right]=\sum_{a, b} E\left[z_{a, b}\right] L_{a, b}=w_{a, b} L_{a, b}=L_{G} \tag{1}
\end{equation*}
$$

NOTE: That

$$
E\left[L_{H}\right]=L_{G} \nRightarrow L_{H} \preccurlyeq(1+\epsilon) L_{G}
$$

Now let's look at sparsifying of H
we want to prove that

$$
\begin{gathered}
L_{H} \preccurlyeq(1+\epsilon) L_{G} \\
L_{G}^{\frac{-1}{2}} L_{H} L_{G}^{\frac{-1}{2}} \preccurlyeq(1+\epsilon) \prod
\end{gathered}
$$

where $\Pi$ is the projection onto the range of $L_{G}$. and $L_{G}^{\frac{-1}{2}}$ denotes square root of of the pseudo-inverse of $L_{G}$

More formally :
Assume that $L_{G}$ isinvertible.

$$
\begin{gathered}
L_{G}^{\frac{-1}{2}} L_{H} \preccurlyeq(1+\epsilon) L_{G}^{\frac{+1}{2}} \\
L_{G}^{\frac{-1}{2}} L_{H} L_{G}^{\frac{-1}{2}} \preccurlyeq(1+\epsilon) I
\end{gathered}
$$

Now Lets look at Expected number of edges $=\sum p_{a, b} \leq \sum\left(\frac{c * l o g(n)}{\epsilon^{2}}\right) q_{a, b}$

$$
\begin{equation*}
q_{a, b} w_{a, b}\left(\delta_{a}-\delta_{b}\right)^{T} L_{G}\left(\delta_{a}-\delta_{b}\right) \tag{2}
\end{equation*}
$$

## Lemma 1.

$$
\sum_{e=(a, b)} q_{a, b}=\sum_{a, b} w_{a, b}\left(\delta_{a}-\delta_{b}\right)^{T} L_{G}^{-1}\left(\delta_{a}-\delta_{b}\right)=\operatorname{Tr}\left(\prod\right)=n-1
$$

Expected number of edges $\leq \frac{c * n \log (n)}{\epsilon^{2}}$
applying Chernoff Bound $\Longrightarrow \operatorname{Pr}\left[\right.$ numberofedges $\left.>\frac{10 * c * n * \operatorname{log(n)}}{\epsilon}\right] \leq 0.0001$ call this expression sparsify.

Defining

$$
\begin{gathered}
X_{a, b}=\left\{\begin{array}{cc}
\frac{w_{a, b}}{p_{a, b}} L_{G}^{\frac{-1}{2}} L_{a, b} L_{G} & \text { with probability } p_{a, b} \\
0 & \text { otherwise }
\end{array}\right. \\
E\left[\sum X_{a, b}\right]=L_{G}^{\frac{-1}{2}} E\left[L_{H}\right] L_{G}^{\frac{-1}{2}}=\prod \\
L_{H}=\sum Y_{a, b}
\end{gathered}
$$

where

$$
\begin{gathered}
Y_{a, b}=\left\{\begin{array}{cc}
\frac{w_{a, b}}{p_{a, b}} L_{a, b} L_{G} & \text { with probability } p_{a, b} \\
0 & \text { otherwise }
\end{array}\right. \\
X_{a, b}=L_{G}^{\frac{-1}{2}} Y L_{G}^{\frac{-1}{2}}
\end{gathered}
$$

Theorem 2. (Joel A Tropp) Let $X_{1}, X_{2}, X_{3}, \ldots . X_{m}$ are matrices of order $n$ and independent s.t $X_{i} \succcurlyeq 0$ random variables. Suppose it satisfy $\left\|X_{i}\right\|<R$

Define $X=\sum_{i=1}^{n}$ So $E[X]=\sum_{i=1}^{n} E\left[X_{i}\right]$ then
$\operatorname{Pr}\left[\lambda_{\min }(X) \leq(1-\epsilon) \mu_{\text {min }}\right]<$ small
$\operatorname{Pr}\left[\lambda_{\max }(X) \geq(1+\epsilon) \mu_{\max }\right]<$ small
where small $\equiv n e^{\frac{-\epsilon^{2} \mu^{\max }}{R}-\epsilon^{2}}$

## Lemma 3.

$$
\left\|X_{a, b}\right\| \leq \operatorname{tr}\left(X_{a, b}\right) \leq \frac{\epsilon^{2}}{c * \log (n)}
$$

where $\left|\left|X_{a, b}\right|\right.$ dentes spectral norm of the matrix.
Analysis:

$$
\begin{gathered}
\sum_{a, b} E\left[X_{a, b}\right]=\prod \\
\| X_{a, b} \left\lvert\, \leq \frac{\epsilon^{2}}{c * \log (n)}=R\right. \\
\operatorname{Pr}\left[\sum_{a, b} X_{a, b} \geq(1+\epsilon) \prod\right] \leq n * e^{-c * \log (n)} \leq \frac{1}{n^{2}}
\end{gathered}
$$

for some $c>10$

